MATH5003P MIDTERM EXAM, FALL 2024

This is an open-book exam, but electronic devices and collaborations are NOT allowed. There are 10 problems, 10 points for each, 100 points in total. BONUS points will be added to the total points. Please provide enough details to justify your answers. The exam time is November 25, from 19:30 to 22:00.

NAME________________ **STUDENT ID.** ________________

Problem 1. Consider manifold $\mathbb{R}^2_{>0}$ and $\varphi : \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$ defined by

F = \varphi $$F(x,y) = \left(xy, \frac{y}{x}\right).$$

(a) [5 points] For vector fields $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $Y = y\frac{\partial}{\partial x}$, compute the pushfoward $\varphi_* X$ and $\varphi_* Y$.
(b) [5 points] Compute $\varphi_*([X,Y])$. Determine whether $\varphi_*([X,Y]) = [\varphi_* X, \varphi_* Y]$ and please justify your answer.

Problem 2. Consider function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$(x,y,z) \mapsto F(x,y,z) := \Big(y^2 + x(x-1)^2(x-2)\Big)^2 + z^2.$$

(a) [7 points] Prove that when $\epsilon > 0$ is sufficiently small, the pre-image $F^{-1}(\epsilon)$ is an embedded submanifold of dimension 2 in $\mathbb{R}^3$.
(b) [3 points] Compute the tangent space of the submanifold $F^{-1}(\epsilon)$ at point $(0,0,\sqrt{\epsilon})$, that is, $T_{(0,0,\sqrt{\epsilon})}F^{-1}(\epsilon)$.

Problem 3. Complete the following three questions about S^3.
(a) [2 points] Write down the definition of a Lie group.
(b) [3 points] Prove, with details, that

$$S^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

is a Lie group.
(c) [2 points] View S^3 as an embedded submanifold in $\mathbb{R}^4$. Compute the normal bundle of S^3 in $\mathbb{R}^4$, that is, $\nu_{S^3}\mathbb{R}^4$.
(d) [3 points] Construct a vector field on S^3 above that does not admit any zeros.

Problem 4. Complete the following four questions about Constant Rank Theorem.
(a) [2 points] State the Constant Rank Theorem. Use it to prove that any submersion is an open map.
(b) [3 points] Let N be a non-empty smooth compact manifold. Prove that there is no smooth submersion $F : N \to \mathbb{R}^k$ for any $k > 0$.
(c) [2 points] State the Global Constant Rank Theorem.
(d) [3 points] Prove that if $\varphi : G \to H$ is a smooth map between two Lie groups which is a group isomorphism, then φ is a smooth diffeomorphism (in particular, its inverse is also smooth).

Problem 5. Complete the following three questions about Lie derivative.
(a) [2 points] State Cartan's magic formula.
(b) [3 points] Write down the definition of a Hamiltonian vector field X_H generated by a function $H : M \to \mathbb{R}$ with respect to a non-degenerate closed 2-form ω (recall that such a 2-form ω is called a symplectic structure on M).
(c) [5 points] Use (a), (b) and the definition of Lie derivative to prove that any element/diffeomorphism in the 1-parameter group of X_H, denoted by φ_H^t, is a *symplectomorphism* in the sense that $(\varphi_H^t)^*\omega = \omega$.

Problem 6. Complete the following questions about Whitney embedding theorem.
(a) [4 points] State Whitney embedding theorem and strong Whitney embedding theorem.
(b) [6 points] Prove the following "compact" version of Whitney embedding theorem: for any compact manifold M, there exists an embedding $M \hookrightarrow \mathbb{R}^N$ when N is sufficiently large.

Problem 7. Complete the following questions about tensors.
(a) [5 points] Recall that an element $x \in V \otimes W$ is called decomposable if there exist $v \in V$ and $w \in W$ such that $x = v \otimes w$. Suppose V admits a basis $\{e_1, e_2\}$ and W admits a basis $\{f_1, f_2, f_3\}$. Determine if

$$x = \sum_{1 \leq i \leq 2,\, 1 \leq j \leq 3} (2024 \cdot i + 5003 \cdot j)(e_i \otimes f_j) \in V \otimes W$$

is decomposable or not. Please justify your answer.
(b) [5 points] Recall that $\Sigma^k V^*$ denotes the symmetric elements in $(V^*)^{\otimes k}$ and $\bigwedge^k V^*$ denotes the alternating elements in $(V^*)^{\otimes k}$. Provide an example $x \in (\mathbb{R}^3)^{*,\otimes 3} = (\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^*$ (where $V = \mathbb{R}^3$) that can *not* be written as

$$x = a + b \quad \text{where } a \in \Sigma^3(\mathbb{R}^3)^* \text{ and } b \in \textstyle\bigwedge^3(\mathbb{R}^3)^*.$$

Please justify your answer.

Problem 8. Complete the following questions about integration.
(a) [2 points] State Stokes' Theorem for a manifold with boundary.
(b) [3 points] Calculate the following integration: Set $S^2 := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ and consider 2-form $\theta = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ restricted on S^2. Compute $\int_{S^2} \theta$.
(c) [5 points] Apply Stokes' Theorem to prove the proposition: Let M be an orientable compact manifold with boundary. Then there does *not* exist any smooth map $F : M \to \partial M$ such that $F|_{\partial M} : \partial M \to \partial M$ is the identity.

Problem 9. Go through the following steps to prove the statement: Let M be a non-empty *compact* smooth manifold of dimension at least 1. If $F : M \to \mathbb{R}$ is a smooth real-valued function, then F has at least two critical points.
(a) [3 points] Show that there exist points $p, q \in M$ such that

$$F(p) = \max_M F \quad \text{and} \quad F(q) = \min_M F.$$

(b) [4 points] Show that p, q from (a) right above must be critical points of F.
(c) [3 points] Prove that F has at least two critical points. (Note that p, q from (b) right above may *not* be different!)

Problem 10. Let $A \in \mathrm{GL}(n + 1, \mathbb{C})$.
(a) [3 points] Show that A induces a well-defined smooth map $\bar{A} : \mathbb{C}P^n \to \mathbb{C}P^n$.
(b) [3 points] Show that the fixed points of $\bar{A} : \mathbb{C}P^n \to \mathbb{C}P^n$ correspond to eigenvectors for the original matrix $A \in \mathrm{GL}(n + 1, \mathbb{C})$.
(c) [4 points] Recall that a smooth map $F : M \to M$ is called a Lefschetz map if for any of its fixed points $p \in M$, the pushforward $\mathrm{d}F(p)$ does *not* have 1 as an eigenvalue. Prove that if eigenvalues of A all have multiplicity 1, then $\bar{A} : \mathbb{C}P^n \to \mathbb{C}P^n$ is a Lefschetz map.
(d) [BONUS 5 points] Recall that when M is a compact manifold and $F : M \to M$ is a Lefschetz map, the Lefschetz number of F is defined by

$$L(F) := \sum_{\text{fixed point } p \text{ of } F} \mathrm{sign}(\det(\mathrm{d}F(p) - \mathbb{1})).$$

Here, "sign" means that if $\det(\mathrm{d}F(p) - \mathbb{1}) > 0$, then sign $= +1$ and if $\det(\mathrm{d}F(p) - \mathbb{1}) < 0$, then sign $= -1$. Under the same condition as in (c) right above, compute the Lefschetz number of $\bar{A} : \mathbb{C}P^n \to \mathbb{C}P^n$.